1 SBA Sample Project A: Analytical Investigation

Project Title:

To assess which solid requires the minimum amount of packaging to create a 250 cm^3 container, between the square based cuboid and cylinder.

Purpose of Project / Problem Statement:

A juice factory wishes to design 250 cm^3 containers for a new line of products. The designers decide one of two ergonomic shapes to package their product; either a square based cuboid or a cylinder. To optimize profits, they decide to choose the shape based on the possibility that it minimizes the amount of material necessary to create each container. The aim of this project is to utilize the principles of calculus and functions, to determine which shape better suits the manufacturer's requirements.

To decide on the shape that will minimize the packaging material used, we must perform the following tasks:

- Determine the minimum surface area of a square-based cuboid whose volume is 250 cm^3.
- Determine the minimum surface area of a cylinder whose volume is 250 cm^3.
- Choose the shape which provides the minimum surface area.

Mathematical Formulation (Cuboid):

Manipulated Variables:

- x: Side of base (cm)
- h: Height (cm)

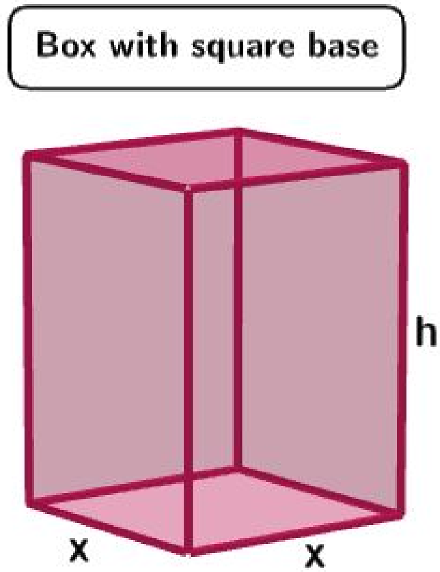
Responding Variable:

- A : Surface area of cuboid (cm^2)

Controlled Variable:

- V : Volume of cuboid (cm^3)

To help visualize the problem, we illustrate with the diagram below.



Volume of cuboid = Area of square base $\times$ Height.

Thus, $V = x^2h$. (i)

Surface area, $A(x,h) = 2x^2 + 4xh$. (ii)

Now, $V = 250$ cm^3.

Thus, $x^2h = 250 \Rightarrow h = \dfrac{250}{x^2}$. (iii)

Substituting (iii) into (ii): $A(x) = 2x^2 + 4x\left(\dfrac{250}{x^2}\right) = 2x^2 + \dfrac{1000}{x}$. (iv)

We must determine the value of x that minimizes the surface area A.

Two methods will be utilized to solve each problem:

1) An analytical method, which involves the computation of the minimum value using the principles of Calculus.

2) A numerical method, whereby a table of values is constructed from the relevant Surface Area function, and the minimum value from each table is extracted via observation.

Solution (Cuboid):

Method 1: Analytical

$$A(x) = 2x^2 + 4x\left(\frac{250}{x^2}\right) = 2x^2 + \frac{1000}{x}.$$

$$\frac{dA}{dx} = \frac{d}{dx}\left(2x^2 + \frac{1000}{x}\right) = 4x - \frac{1000}{x^2}.$$

For stationary value, $\dfrac{dA}{dx} = 0$

$$\Rightarrow 4x - \frac{1000}{x^2} = 0$$
$$\Rightarrow 4x^3 - 1000 = 0$$
$$\Rightarrow x = \sqrt[3]{250} = 6.30 \text{ cm}$$

When $x = 6.30$ cm, $h = \dfrac{250}{(6.30)^2} = 6.30$ cm.

We must verify that A is indeed a minimum value.

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{dA}{dx}\right) = \frac{d}{dx}\left(4x - \frac{1000}{x^2}\right) = \frac{d}{dx}\left(4x - 1000x^{-2}\right) = 4 + 2000x^{-3} = 4 + \frac{2000}{x^3}.$$

When $x = 6.30$, $\dfrac{d^2A}{dx^2} = 4 + \dfrac{2000}{(6.30)^3} = 12 > 0$.
Thus, $\frac{d^2A}{dx^2} > 0$ and the surface area of the square based cuboid attains a minimum value when $x = 6.30$ cm.
Now, when $x = 6.30$ cm, $A = 2(6.30)^2 + \dfrac{1000}{(6.30)} = 238.11 \text{ cm}^2$.

The minimum surface area of the cuboid is 238.11 cm^2, and occurs when the dimensions of the package are $x = h = 6.30$ cm.

Method 2: Numerical

Using Microsoft Excel, tables of values corresponding to the surface area $A(x)$ are generated for varying values of x. This is done in two phases, which facilitate us in locating the value of x for which the minimum occurs, to a fair degree of accuracy. Recall that $A(x) = 2x^2 + \dfrac{1000}{x}$.

- **Phase 1**: Let x take integer values. Thus, we can estimate the location of the turning point to the nearest whole number.

  We use x-increments of 1 for the interval $1 \leq x \leq 10$. The first row of the table below contains the independent variable x, while the dependent surface areas $A(x)$ are in the second row.

x	1	2	3	4	5	**6**	7	8	9	10
$A(x)$	1002	508	351.33	282	250	**238.67**	240.86	253	273.11	300

- From this preliminary table, we observe that the minimum surface area of 238.67 cm^2 occurs when the base has a side of $x = 6$ cm. The minimum value (turning point) will occur in the interval that is 1 unit from $x = 6$; that is in the region $5 \leq x \leq 7$.

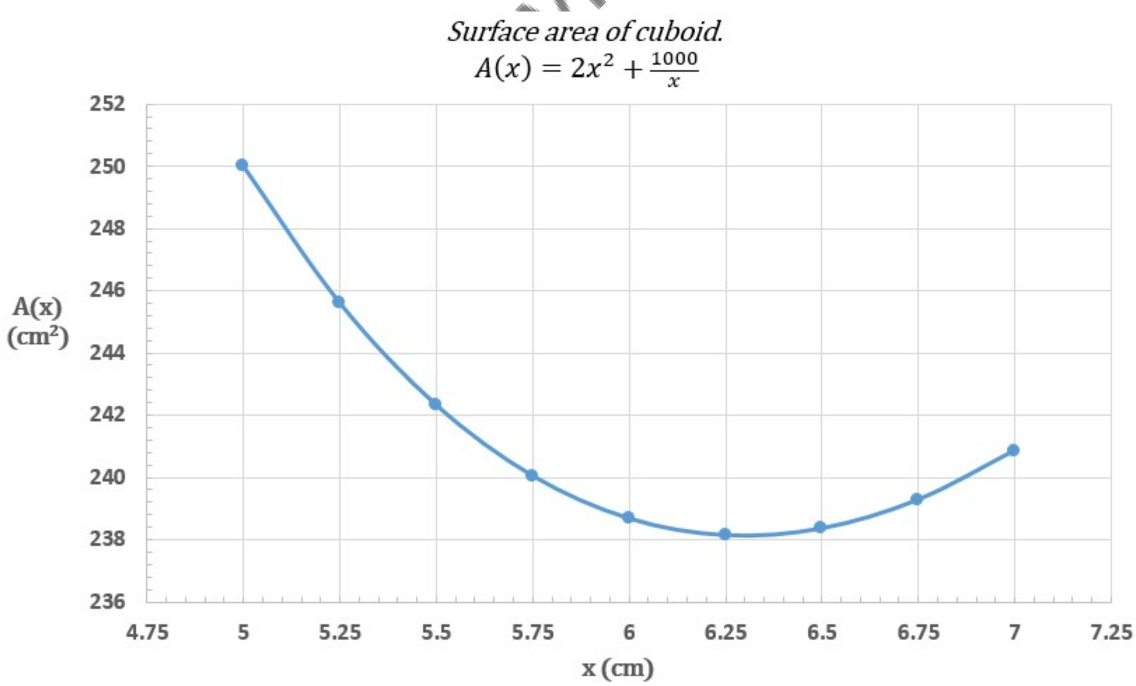
- <u>**Phase 2**</u>: Now that we have narrowed our solution to $5 \leq x \leq 7$, we can use smaller increments across this interval, say of 0.25 to yield a better approximation for the minimum value.

x	5	5.25	5.5	5.75	6	**6.25**	6.5	6.75	7
$A(x)$	250	245.60	242.32	240.04	238.67	**238.13**	238.35	239.27	240.86

- From this secondary table, a better estimate of the minimum surface area is 238.13 cm^2, which occurs when $x = 6.25$ cm.

When $x = 6.25$ cm, $h = \dfrac{250}{(6.25)^2} = 6.40$ cm.

From the value given in the previous table, the associated graph (plotted in Microsoft Excel) shows that the minimum value occurs in the vicinity of $x = 6.25$ cm.



Mathematical Formulation (Cylinder):

Manipulated Variables:

- r: Radius of cylinder (cm)
- h: Height of cylinder (cm)

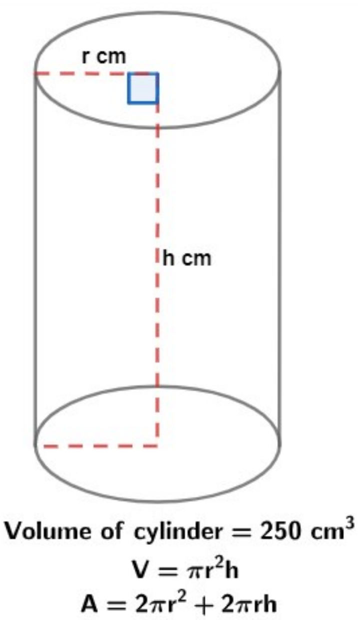
Responding Variable:

- A : Surface area of cylinder (cm^2)

Controlled Variable:

- V : Volume of cylinder (cm^3)

A relevant diagram of the cylinder is shown below.



The governing equations:

$V = \pi r^2 h.$

$A = 2\pi r^2 + 2\pi r h.$

Since $V = 250$, this implies that $\pi r^2 h = 250$.

Now $A = 2\pi r^2 + 2\pi r h$.

Also, $\pi r^2 h = 250 \Rightarrow h = \dfrac{250}{\pi r^2}$.

Thus, $A = 2\pi r^2 + 2\pi r \left(\dfrac{250}{\pi r^2}\right) = 2\pi r^2 + \dfrac{500}{r}$.

We must determine the value of r that minimizes the surface area A.

Solution (Cylinder):

Method 1: Analytical

$A = 2\pi r^2 + \dfrac{500}{r}.$

$\dfrac{dA}{dr} = 4\pi r - \dfrac{500}{r^2}.$

To minimize area, $\dfrac{dA}{dr} = 0.$

$\Rightarrow 4\pi r^3 - 500 = 0$

$\Rightarrow r = \sqrt[3]{\dfrac{500}{4\pi}} = 3.41$ cm.

When $r = 3.41$, $h = \dfrac{250}{\pi(3.41)^2} = 6.84$ cm.

$A = 2\pi(3.41)^2 + \dfrac{500}{3.41} = 219.69 \text{ cm}^2.$

Has the surface area been minimized?

The second derivative of A is $\dfrac{d^2A}{dr^2} = 4\pi + \dfrac{1000}{r^3}.$

When $r = 3.41$, $\dfrac{d^2A}{dr^2} = 4\pi + \dfrac{1000}{(3.41)^3} = 37.79 > 0.$

Therefore A is a minimum when $r = 3.41$.

The dimensions of the cylindrical package are $r = 3.41$ cm and $h = 6.84$ cm. The minimum possible surface area of the required cylinder is 219.69 cm^2.

Method 2: Numerical

Using Microsoft Excel, tables of values corresponding to the surface area $A(r)$ are generated for varying values of radius, r. As done previously, we facilitate our estimates in two phases; to the nearest integer, and then to the nearest 0.25. Recall that $A = 2\pi r^2 + \dfrac{500}{r}$.

- **Phase 1**:

  We use r-increments of 1 for the interval $1 \le r \le 10$.

r	1	2	**3**	4	5	6	7	8	9	10
$A(r)$	506.28	278.13	**223.22**	225.53	257.08	309.53	379.31	464.62	564.49	678.32

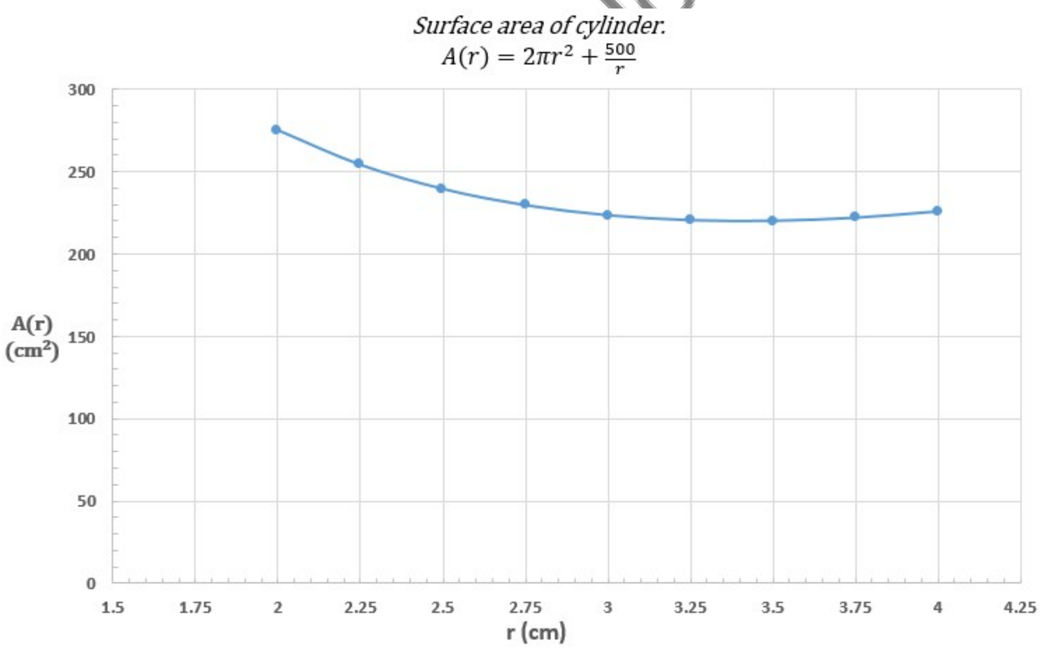
- From this preliminary table, we observe that the minimum surface area of 223.22 cm^2 occurs when the radius of the cylinder, $r = 3$ cm. The actual minimum will occur either to the left or the right of $r = 3$; that is in the region $2 \leq r \leq 4$.

- **<u>Phase 2</u>**: Use increments of 0.25 for $2 \leq r \leq 4$.

r	2	2.25	2.5	2.75	3	3.25	**3.5**	3.75	4
$A(r)$	275.13	254.03	239.27	229.34	223.22	220.21	**219.83**	221.69	225.53

- From this secondary table, a better estimate of the minimum surface area is 219.83 cm^2, which occurs when $r = 3.5$ cm.

When $r = 3.5$, $h = \dfrac{250}{\pi(3.5)^2} = 6.5$ cm.
Using the values from the previous table, the graph of Area of cylinder versus Radius of cylinder was plotted. It reveals that the minimum value of the area function occurs in the vicinity of $r = 3.5$ cm.



Application of Solution:

Based on the analytical method, the minimum surface area of the square-based cuboid was calculated to be 238.11 cm^2, with dimensions 6.30 cm $\times$ 6.30 cm $\times$ 6.30 cm, while with the numerical method, the minimum surface area of the cuboid was 238.13 cm^2, with the dimensions 6.25 cm $\times$ 6.25 cm $\times$ 6.40 cm.

For the cylinder, the analytical method yielded a minimum surface area of 219.69 cm^2, with radius 3.41 cm and height 6.84 cm. Using the numerical method, the minimum surface area of the cylinder was calculated to be 219.83 cm^2, with a radius of 3.50 cm and a height of 6.50 cm.

Discussion of Findings:

The analytical procedure, which is predicated on the principles of calculus, is more efficient and yields more accurate results than its numerical counterpart. At best, a table of values generates a good estimate, while the analytical method provides results which are more reliable and trustworthy. However, a student can only apply this method if he is knowledgeable in calculus, particularly with the concept of the derivative. One that is deficient in this knowledge will have to solve the problems numerically, which is a less sophisticated and more tedious method. Without the use of mathematical software, such as Microsoft Excel, the numerical method can be extremely time consuming. Using the software, we were able to construct two tables to get a more reliable estimate of the minimum surface area; the first with step increments of 1 cm, and the second with increments of 0.25 cm. However, if the calculator alone was at our disposal, perhaps only one table would have been constructed. In such a case, the dimensions of the required solids would have been estimated at best to the nearest centimetre, which of course is relatively inaccurate compared to the actual results.

One of the assumptions made was in computing the surface area for an exact volume of 250 cm^3. In reality, the package should be made slightly larger than the volume that will contain, to avoid spillage and to ensure that the package and contents can be adequately sealed.

Conclusion:

The purpose of this project was to determine the shape with the minimum surface area that will contain a volume of 250 cm^3. The calculations show that the cylinder can be created to house the same volume as the cuboid, but by using a smaller surface area. Thus, the cylinder is the preferable shape for packaging, and this information can be used by a manufacturer to assist in minimizing costs associated with production.

In future studies, perhaps a comparison with other shapes and their surface areas can be made. However, it is unclear on the utility of such studies, since some packages may be more difficult to create. Thus, although a specific shape may minimize surface area, the overall costs with manufacturing that shape (such as in welding, for instance), may make it unsuitable for mass packaging. On another note, shapes such as the tetrahedron may not be as well suited for shipping purposes, as cuboids and cylinders may be more comfortably fitted into a shipping vessel, when distribution is actually being undertaken. Topics for future consideration include minimizing the surface area of sports equipment such as soccer and volleyballs, and thus alternate applications are not necessarily restricted to beverage manufacturing only.